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The contact problem for a rectangle with stress-free side faces☆

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Abstract

The plane contact problem for an elastic rectangle into which two symmetrically positioned punches are impressed is considered. Homogeneous solutions are constructed that leave the side faces of the rectangle stress-free. When the modified boundary conditions using generalized orthogonality of the homogeneous solutions are satisfied, the problem reduces to a Friedholm integral equation of the first kind in the function describing the displacement of the surface of the rectangle outside the contact area. This function is sought in the form of the sum of a trigonometric series and a power function with a root singularity. The ill-posed infinite system of algebraic equations thereby obtained is regularized by introducing a small positive parameter (Ref. Kalitkin NN. Numerical Methods. Moscow: Nauka; 1978), and, after reduction, has a stable regularized solution. Since the matrix elements of the system are determined by a poorly converging number series, an effective method was developed for calculating the residues of the series. Formulae are found for the contact pressure distribution function and dimensionless indentation force. Since the first formula contains a third-order derivative of the functional series, when it is used, a numerical differentiation procedure is employed (Refs. Kalitkin NN. Numerical Methods. Moscow: Nauka; 1978; Danilina NI, Dubrovskaya NS, Kvasha OP et al. Numerical Methods. Textbook for Special Colleges. Moscow: Vysshaya Shkola; 1976). Examples of a calculation for a plane punch are given.

1. Formulation of the problem and homogeneous solutions

In Cartesian coordinates x, y, we consider the problem of the indentation into an elastic rectangle of two identical and symmetrically positioned punches having a width $2a$ and bases $y = \pm(b - \delta(x))$, where $\delta(x)$ is an even function in x (Fig. 1).

We will assume that in the contact area of the punch and rectangle there are no friction forces, and outside the contact area there is no adjoining load. Then the boundary conditions can be written in the form

$$\sigma_x(\pm 1, y) = \tau_{xy}(\pm 1, y) = 0, \quad |y| \le b \tag{1.1}$$

$$\tau_{xy}(x, \pm b) = 0, \ |x| \le 1; \ \upsilon(x, \pm b) = \mp\delta(x), \ |x| \le a; \ \sigma_y(x, \pm b) = 0, \ a < |x| \le 1 \tag{1.2}$$

where u and υ are components of the vector of displacement of points of the elastic medium.

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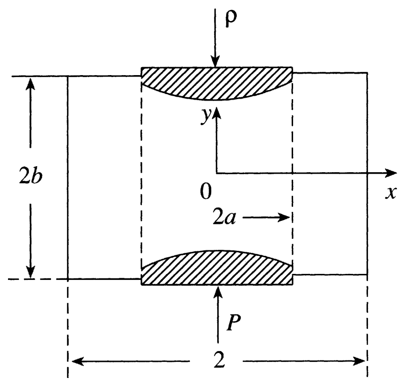

Fig. 1.

We will solve the problem of plane strain in terms of stresses, expressing the quantities required in terms of the biharmonic function Φ:

$$\Delta^2\Phi = 0, \quad \sigma_x = \partial^2\Phi/\partial y^2, \quad \sigma_y = \partial^2\Phi/\partial x^2, \quad \tau_{xy} = -\partial^2\Phi/\partial x\partial y$$

Searching for the stress function in the form $\Phi = \varphi(x)\cos zy$ ($z = \text{const}$), we find

$$\begin{aligned} &\varphi(x) = C_1 \operatorname{ch} zx + C_2 x \operatorname{sh} zx \\ &\sigma_x = -\varphi(x)z^2\cos zy, \quad \tau_{xy} = -\varphi'(x)z\sin zy, \quad \sigma_y = \varphi''(x)\cos zy \end{aligned} \tag{1.3}$$

Boundary conditions (1.1) will be satisfied if the function $\varphi(x)$ obeys the relations

$$\varphi(\pm 1) = C_1 \operatorname{ch} z + C_2 \operatorname{sh} z = 0, \quad \varphi'(\pm 1) = C_1 z \operatorname{sh} z + C_2(\operatorname{sh} z + z\operatorname{ch} z) = 0$$

Non-trivial solutions of this system are related to the roots z_n of the equation

$$\operatorname{sh} 2z + 2z = 0 \quad (\operatorname{Re} z_n \geq 0;\ n = 0, 1, \ldots) \tag{1.4}$$

The asymptotic form of these roots and an iteration scheme for calculating them are well-known.[3,4]

Assuming in Eq. (1.3) that

$$C_1 = -\beta_n \operatorname{th} z_n/2, \quad C_2 = \beta_n/2, \quad \beta_n = (z_n \operatorname{ch} z_n)^{-1}$$

we obtain the eigenfunction $F_n(x)$ and the stress state corresponding to the non-zero eigenvalue z_n

$$\begin{aligned} &F_n(x) = \frac{1}{2}(x\operatorname{sh} z_n x - \operatorname{ch} z_n x \operatorname{th} z_n)\beta_n, \quad F''_n(x) = F_n(x)z_n^2 + \frac{\operatorname{ch} z_n x}{\operatorname{ch} z_n}, \quad \Phi_n = b_n F_n(x)\cos z_n y \\ &\sigma_x^{(n)} = -b_n F_n(x) z_n^2 \cos z_n y, \quad \sigma_y^{(n)} = b_n F''_n(x)\cos z_n y, \quad \tau_{xy}^{(n)} = b_n F'_n(x) z_n \sin z_n y \\ &b_n = \text{const}; \quad n = 1, 2, \ldots \end{aligned} \tag{1.5}$$

The following equations correspond to the root z_0

$$\Phi_0 = b_0 F_0(x) = b_0 x^2/2, \quad \sigma_x^{(0)} = \tau_{xy}^{(0)} = 0, \quad \sigma_y^{(0)} = b_0 \tag{1.6}$$

Substituting expressions (1.5) and (1.6) into the relations of Hooke's law and integrating them, we find the components of displacement

$$2\theta u^{(0)} = \nu_0 b_0 x, \quad 2\theta u^{(n)} = b_n(F_n'''(x)/z_n^2 - \nu_1 F_n'(x))\cos z_n y$$
$$2\theta \upsilon^{(0)} = b_0 y, \quad 2\theta \upsilon^{(n)} = b_n(F_n''(x)/z_n - \nu_0 z_n F_n(x))\sin z_n y \tag{1.7}$$
$$\theta = G/(1-\nu), \quad \nu_0 = \nu/(\nu-1), \quad \nu_1 = (2-\nu)/(1-\nu)$$

where G is the shear modulus and ν is Poisson's ratio.

From relations (1.5)–(1.7) with $y = b$ we find

$$\sigma_y^{(0)}(x) = b^{-1} f_0, \quad \sigma_y^{(n)}(x) = f_n F_n''(x) z_n \operatorname{ctg}(z_n b), \quad \tau_{xy}^{(n)}(x) = f_n F_n'(x) z_n^2$$
$$2\theta u^{(0)}(x) = \nu_0 b^{-1} f_0 x, \quad 2\theta u^{(n)}(x) = f_n(F_n'''(x)/z_n - \nu_1 z_n F_n'(x))\operatorname{ctg}(z_n b)$$
$$2\theta \upsilon^{(0)}(x) = f_0, \quad 2\theta \upsilon^{(n)}(x) = f_n(F_n''(x) - \nu_0 F_n(x) z_n^2) \tag{1.8}$$
$$f_0 = b_0 b, \quad f_n = b_n \sin(z_n b)/z_n; \quad n = 1, 2, \dots$$

For homogeneous solutions (1.8) corresponding to two different roots z_n and z_m ($n \neq m$), Betti's work reciprocity theorem can be written in the form

$$J(z_n, z_m) - J(z_m, z_n) = 0, \quad n \neq m \tag{1.9}$$

$$J(z_m, z_n) = \int_{-1}^{1} [\sigma_y^{(m)}(x)\upsilon^{(n)}(x) - \tau_{xy}^{(n)}(x) u^{(m)}(x)]dx \tag{1.10}$$

Substituting expressions (1.8) into Eq. (1.10) and integrating by parts, we obtain

$$J(z_m, z_n) = -(2\theta)^{-1} f_m f_n \operatorname{ctg}(z_m b) z_m^3 J^*(z_m, z_n), \quad m \neq n = 1, 2, \dots$$
$$J^*(z_m, z_n) = \int_{-1}^{1} [F_n'(x)\beta_m \operatorname{sh} z_m x + F_m'(x)\beta_n \operatorname{sh} z_n x]dx \tag{1.11}$$

From relations (1.9) and (1.11) we find the condition of generalized orthogonality of the eigenfunctions $F_n(x)$ ($n = 1, 2, \dots$)

$$J^*(z_m, z_n) = \begin{cases} 0, & m \neq n \\ -z_n^{-2}, & m = n \end{cases} \tag{1.12}$$

2. Method of solution

We will use a well-known approach[5] and redefine the function $\upsilon(x, b)$, assuming that

$$\upsilon(x, b) = \tilde{\upsilon}(x) = \begin{cases} -\delta(x), & |x| \leq a \\ -g(x), & a \leq |x| \leq 1 \end{cases} \tag{2.1}$$

where $g(x)$ is the required function, which is even in x. Then the second boundary condition (1.2) can be written in the form

$$\upsilon(x, b) = \tilde{\upsilon}(x), \quad |x| \leq 1 \tag{2.2}$$

We will introduce an abridged notation of the sum

$$\sum_{n=h_0}^{\infty}{}' G_n(x) = 2\mathrm{Re}\left\{\sum_{n=h_0}^{\infty} G_n(x)\right\} \quad (\mathrm{Re}z_n, \mathrm{Im}z_n > 0)$$

Since the functional series

$$\tau_{xy}(x, b) = \sum_{n=1}^{\infty}{}' \tau_{xy}^{(n)}(x), \quad \upsilon(x, b) = \sum_{n=0}^{\infty}{}' \upsilon^{(n)}(x), \quad \sigma_y(x, b) = \sum_{n=0}^{\infty}{}' \sigma_y^{(n)}(x)$$

determining the left-hand sides of the first and final conditions of system (1.2) and conditions (2.2) differ (as indicated by an *a posteriori* analysis of the solution), the boundary conditions are replaced by the following

$$\int_0^x\int_0^\eta \tau_{xy}(\xi, b)d\xi d\eta \equiv \sum_{n=1}^{\infty}{}' f_n(F_n'(x) - \beta_n \mathrm{sh}z_n x) = 0, \quad |x| \le 1 \tag{2.3}$$

$$2\theta\int_0^x \upsilon(\xi, b)d\xi \equiv f_0 x + \sum_{n=1}^{\infty}{}' f_n[F_n'(x) - \nu_0(F_n'(x) - \beta_n \mathrm{sh}z_n x)] = 2\theta\int_0^x \tilde{\upsilon}(\xi)d\xi \tag{2.4}$$

$$\sigma(x) \equiv \int_1^x\int_1^t\int_1^\eta \sigma_y(\xi, b)d\xi d\eta dt \equiv \frac{1}{2}f_0 f(x) + \frac{1}{4}\sum_{n=1}^{\infty}{}' f_n \tilde{F}_n(x) = 0, \quad a \le x \le 1$$

$$f(x) = \frac{(x-1)^2}{3b}, \quad \tilde{F}_n(x) = 4z_n \mathrm{ctg}z_n b\int_1^x F_n(t)dt = 4\mathrm{ctg}z_n b\left[\frac{F_n'(x)}{z_n} + \frac{\mathrm{th}z_n - \mathrm{sh}z_n x/\mathrm{ch}z_n}{z_n^2}\right] \tag{2.5}$$

Eqs. (2.3) and (2.4) are equivalent to the system of relations

$$\sum_{n=1}^{\infty}{}' f_n F_n'(x) = 2\theta\int_0^x \tilde{\upsilon}(\xi)d\xi - f_0 x, \quad \sum_{n=1}^{\infty}{}' f_n \beta_n \mathrm{sh}z_n x = 2\theta\int_0^x \tilde{\upsilon}(\xi)d\xi - f_0 x, \quad |x| \le 1 \tag{2.6}$$

Here

$$f_0 = 2\theta\int_0^1 \tilde{\upsilon}(\xi)d\xi \tag{2.7}$$

Now, using the condition of generalized orthogonality (1.12), we determine the constants f_n. Multiplying the first equation of system (2.6) by $\beta_m x$, and the second equation by $F_m'(x)$, and then summing and integrating with respect to x, we obtains

$$f_n = 4\theta\int_0^1 \tilde{\upsilon}(\xi)F_n''(\xi)d\xi, \quad n = 1, 2, \ldots \tag{2.8}$$

Replacing the coefficients $f_0, f_1, f_2, \ldots$ in relation (2.5) by integrals (2.7) and (2.8), and taking into account Eqs. (2.1), condition (2.5) takes the form

$$\sigma(x) \equiv -\theta\left\{\int_a^1 g(\xi)K(\xi, x)d\xi + \int_0^a \delta(\xi)K(\xi, x)d\xi\right\} = 0, \quad a \le x \le 1 \tag{2.9}$$

where

$$K(\xi, x) = f(x) + {\sum_{n=1}^{p}}' F_n''(\xi)\tilde{F}_n(x) + R_p(\xi, x), \quad R_p(\xi, x) = {\sum_{n=p+1}^{\infty}}' F_n''(\xi)\tilde{F}_n(x)$$

$$p \ge 4000$$

Let the prescribed $\delta(\xi)$ and the required $g(\xi)$ functions be defined by the series

$$\delta(\xi) = \sum_{k=0}^{\infty} \delta_k \cos a_k\xi, \quad 0 \le \xi \le a, \quad a_k = \frac{k\pi}{a}; \quad g(\xi) = \sum_{k=0}^{\infty} \delta_k g_k(\xi), \quad a \le \xi \le 1$$
$$g_k(\xi) = X_*^{(k)} + \sum_{j=0}^{1} X_j^{(k)}(\xi - a)^{(j+1)/2} - \sum_{r=1}^{\infty} X_{r+1}^{(k)} l_r^{-2}\cos l_r(\xi - a), \quad l_r = \frac{r\pi}{l}, \quad l = 1 - a \tag{2.10}$$

From the condition $\delta(a) = g(a)$ we find

$$X_*^{(k)} = (-1)^k + \sum_{r=1}^{\infty} X_{r+1}^{(k)} l_r^{-2}, \quad k = 0, 1, \ldots \tag{2.11}$$

$$g_k(\xi) = (-1)^k + \sum_{j=0}^{1} X_j^{(k)}(\xi - a)^{(j+1)/2} + \sum_{r=1}^{\infty} X_{r+1}^{(k)} l_r^{-2}(1 - \cos l_r(\xi - a)), \quad a \le \xi \le 1 \tag{2.12}$$

Substituting expressions (2.10) and (2.12) into Eq. (2.9) and equating the coefficients of δ_k ($k = 0, 1, \ldots$) to zero, we obtain a system of functional equations

$$\sum_{s=0}^{\infty} X_s^{(k)}[f_s(x) + j_s f(x)] = f^{(k)}(x) - \varepsilon_k f(x), \quad a \le x \le 1 \tag{2.13}$$

where

$$f_s(x) = {\sum_{n=1}^{p}}' Q_{s,n}\tilde{F}_n(x) + R_p^{(s)}, \quad f^{(k)}(x) = a_k^2 {\sum_{n=1}^{\infty}}' I_{kn}\tilde{F}_n(x) \tag{2.14}$$

$$Q_{0,n} = J_n,\ Q_{1,n} = I_n,\ Q_{r+1,n} = J_{rn};\ j_0 = 2l^{3/2}/3,\ j_1 = l^2/2,\ j_{r+1} = ll_r^{-2}$$

$$p \ge 4000,\quad \varepsilon_0 = 1,\quad \varepsilon_r = (-1)^r l,\quad r = 1, 2, \ldots$$

$$J_n = \int_a^1 \sqrt{\xi - a} F_n''(\xi) d\xi = \frac{\sqrt{l}}{4}\Big[(\beta_n \operatorname{sh} z_n a - 4F_n'(a))C_n -$$

$$-(\beta_n \operatorname{ch} z_n a + 4z_n F_n(a))S_n - \frac{\operatorname{th} z_n}{z_n}\Big],\quad n \le n_0 = \operatorname{entier}(7/l)$$

$$C_n \equiv \frac{C(z_n^*)}{z_n^*} = \sum_{k=0}^{\infty} \frac{(z_n l)^{2k}}{(2k)!(4k+1)},\quad S_n \equiv \frac{iS(z_n^*)}{z_n^*} = \sum_{k=0}^{\infty} \frac{(z_n l)^{2k+1}}{(2k+1)!(4k+3)},\quad z_n^* = \sqrt{\frac{2z_n l}{i\pi}}$$

$$J_n = \frac{l}{8}\sqrt{\frac{\pi}{z_n}}\{\exp(z_n l)[1 + \tilde{z}_n - \operatorname{th} z_n(1 - \tilde{z}_n)] + i\exp(-z_n l)[1 - \tilde{z}_n + \operatorname{th} z_n(1 + \tilde{z}_n)]\} + \tag{2.15}$$

$$+ 2l^{3/2}\left\{\sum_{m=1}^{10} (4m-1)!!m\tilde{z}_n^{2m+2} + \operatorname{th} z_n \sum_{m=0}^{10} (4m+1)!!(m+1)\tilde{z}_n^{2m+3}\right\},\quad n > n_0,\quad \tilde{z}_n = \frac{1}{2z_n l}$$

$$J_{rn} = l_r^{-2}\int_a^1 F_n''(\xi)(1 - \cos l_r(\xi - a))d\xi = \operatorname{sh} z_n \frac{(-1)^r \operatorname{sh} z_n - \operatorname{sh} z_n a}{(l_r^2 + z_n^2)^2} - \frac{F_n'(a)}{l_r^2 + z_n^2},\quad r = 1, 2, \ldots$$

$$I_{kn} = \int_0^a F_n(\xi)\cos a_k \xi d\xi = (-1)^k\left[\frac{\operatorname{sh} z_n \operatorname{sh} z_n a}{(a_k^2 + z_n^2)^2} + \frac{F_n'(a)}{a_k^2 + z_n^2}\right],\quad I_n = \int_a^1 (\xi - a)F_n''(\xi)d\xi = F_n(a)$$

$$\int_0^a F_n''(\xi)\cos a_k \xi d\xi = (-1)^k F_n'(a) - a_k^2 I_{kn}$$

Here, $R_p^{(10)} = R_p^{(0)}(x)$, $R_p^{(1)} = R_p^{(1)}(x)$ and $R_p^{(r+1)} = R_p(r, x)$ are the residues of the functional series (2.14), and $C(z_n^*)$ and $S(z_n^*)$ are Fresnel integrals.[6]

An analysis of the above-mentioned residues (see Section 3) indicates that the series (2.14) converge uniformly in the segment [0, 1], and consequently they can be integrated term by term. Multiplying Eq. (2.13) by $\cos l_m(x - a)$ ($m = 0, 1, \ldots$) and integrating over the segment $[a, 1]$, we obtain an infinite system of algebraic equations in the unknowns $X_s^{(k)}$ ($s = 0, 1, \ldots$)

$$A\mathbf{X}^{(k)} = \mathbf{b}^{(k)},\quad k = 0, 1, \ldots \tag{2.16}$$

Bearing in mind the integrals

$$J_r^0 = \int_a^1 f(x)\cos l_r(x - a)dx = \frac{1}{bl_r^2}\left\{\frac{2[1 - (-1)^r]}{l_r^2} - l^2\right\},\quad J_0^0 = -\frac{l^4}{12b},\quad \tilde{\varepsilon}_0 = l,\quad \tilde{\varepsilon}_r = 0$$

$$\tilde{J}_{mn} = \int_a^1 \tilde{F}_n(x)\cos l_m(x - a)dx = 4z_n \operatorname{ctg}(z_n b)\left\{\tilde{\varepsilon}_m \frac{\operatorname{th} z_n}{z_n^3} - \frac{F_n(a)}{l_m^2 + z_n^2} + \frac{\operatorname{ch} z_n a/\operatorname{ch} z_n - (-1)^m}{(l_m^2 + z_n^2)^2}\right\}$$

we obain the following expressions for the elements of the matrix A and the vector $\mathbf{b}^{(k)}$

$$a_{m,s} = j_s J_m^0 + \sum_{n=1}^{p}{}' Q_{s,n}\tilde{J}_{mn} + R_p^{(s)}, \quad b_m^{(k)} = a_k^2 \sum_{n=1}^{\infty}{}' I_{kn}\tilde{J}_{mn} - \varepsilon_k J_m^0, \quad m, s = 0, 1, \dots$$

where $R_p^{(0)} = R_p^{(0)}(m)$, $R_p^{(1)} = R_p^{(1)}(m)$ and $R_p^{(r+1)} = R_p(r, m)$ are the residues of the corresponding number series, $p \geq 4000$; the values of j_s and $Q_{s,n}$ are given by formulae (2.15).

Integral Eq. (2.9) is the consequence of an ill-posed problem, and therefore system (2.16) is ill-posed defined and must be regularized by introducing the small positive parameter α.[1] The regularized system has the form

$$(A^*A + \alpha E)\mathbf{Y}^{(k)} = A^*\mathbf{b}^{(k)}, \quad k = 0, 1, \dots \tag{2.17}$$

where A^* is the Hermitian conjugate matrix.

Having determined from system (2.17) the regularized solution $\mathbf{Y}^{(k)}$ and the function

$$g_k(\xi) = (-1)^k + \sum_{j=0}^{1} Y_j^{(k)}(\xi - a)^{(j+1)/2} + 2\sum_{r=1}^{\infty} Y_{r+1}^{(k)}[l_r^{-1}\sin(l_r(\xi - a)/2)]^2, \; k = 0, 1, \dots \tag{2.18}$$

by means of formula (2.9) we then find the function $\sigma(x)$, in terms of which the stress $\sigma_y(x, b) = \sigma'''(x)$ is expressed. We have

$$\begin{gathered} \sigma(x) = \theta\sum_{k=0}^{\infty}\delta_k\sigma_k(x), \quad \sigma_k(x) = -\gamma_k f(x) - \omega^{(k)}(x) \\ \gamma_k = \varepsilon_k + \sum_{s=0}^{\infty} Y_s^{(k)} j_s, \quad \omega^{(k)}(x) = \sum_{s=0}^{\infty} Y_s^{(k)} f_s(x) - f^{(k)}(x) \end{gathered} \tag{2.19}$$

Note also that, when $n > \text{entier}(10/b)$, the values of the function $\text{ctg} z_n b$ and the hyperbolic functions, for example $\text{sh} z_n x$, are calculated by means of the formulae

$$\text{ctg} z_n b = -i, \quad \text{sh} z_n x = \text{sh} z(x, n)$$

$$z(x, n) = x(z_n - i\pi n + i\pi/4) + i2\pi \text{mod}[x(n/2 - 1/8), 1]$$

3. Calculation of the residues $R_p(\xi, x), \dots, R_p(r, m)$

If the notation

$$\lambda_n = (-4z_n)^{-1}, \quad z(\theta) = \exp(i\pi\theta), \quad x_r = r/(lp), \quad l_r/z_n = -4\pi p x_r \lambda_n$$

is introduced and Eq. (1.4) is solved for $\exp(2z_n)$, then, raising $\exp(2z_n)$ to the power of a, it is possible to find

$$\exp(2z_n a) = z^n(2a)\lambda_n^{-a}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\lambda_n^2}\right]^a = z^n(2a)\lambda_n^{-a}\left[1 + a\lambda_n^2 + \frac{1}{2}a(a-3)\lambda_n^4 + \dots\right]$$

Taking this formula into account for large n and small $l_r/z_n(|l_r/z_n|<x_r)$, the integrals J_n, I_n, J_{rn} and $\tilde{J}_{mn}$ and the functions $F_n''(\xi)$ and $\tilde{F}_n(x)$ can be expanded into a series in powers of λ_n

$$J_n = \sqrt{\pi}\lambda_n[iq_{0,n}(a)+q_{0,n}(-a)] - 16l^{-3/2}[\lambda_n^3-(2+6/l)\lambda_n^4+\dots]$$

$$I_n = 2\lambda_n^{3/2}[q_{1,n}(-a)+q_{1,n}(a)], \quad 8F_n''(\xi) = \lambda_n^{-1/2}[q_n^*(\xi)+q_n^*(-\xi)]$$

$$i\tilde{F}_n(x) = 8\lambda_n^{3/2}[\tilde{q}_n(-x)-\tilde{q}_n(x)] + 64[\lambda_n^2-2\lambda_n^3+2\lambda_n^4+\dots]$$

$$q_{0,n}(a) = z^n(1-a)\lambda_n^{a/2}[(1+a)/2-\lambda_n(1+a)/2+\lambda_n^2(5-a^2)/4+\dots]$$

$$q_{1,n}(a) = z^n(1-a)\lambda_n^{a/2}[1+a-\lambda_n(3+a)+\lambda_n^2(9-a^2)/2+\dots]$$

$$q_n^*(\xi) = z^n(1-\xi)\lambda_n^{\xi/2}[1+\xi+\lambda_n(5-\xi)-\lambda_n^2(7+\xi^2)/2+\dots]$$

$$\tilde{q}_n(x) = z^n(1-x)\lambda_n^{x/2}[1+x-\lambda_n(7+x)+\lambda_n^2(17-x^2)/2+\dots] \text{ и т.д.}$$

From this, multiplying the expansions of the corresponding quantities and discarding terms of a higher order of smallness than λ_n^2, we find, with the accuracy indicated, expressions for the n-th terms of the residues

$$F_n''(\xi)\tilde{F}_n(x) = Q_{0,n}^*, \quad J_n\tilde{F}_n(x) = Q_{1,n}^*, \quad I_n\tilde{F}_n(x) = Q_{2,n}^* \text{ и т.д.} \tag{3.1}$$

The expressions $Q_{k,n}^*$ are the sums of a finite number of terms of the form $\lambda_n^s A$ or $z^n(\theta)\lambda_n^s A$. In view of the length of these expressions, only the most significant are given below

$$\begin{aligned} iQ_{0,n}^* &= Q_{0,n}(\xi,-x)-Q_{0,n}(\xi,x)+Q_{0,n}(-\xi,-x)-Q_{0,n}(-\xi,x)+8(\tilde{Q}_{0,n}(\xi)+\tilde{Q}_{0,n}(-\xi)) \\ Q_{0,n}(\xi,x) &= z^n(-\xi-x)\lambda_n^{(2+\xi+x)/2}\{(1+\xi)(1+x)+\lambda_n[(1+x)(5-\xi)-(1+\xi)(7+x)]+\dots\} \\ \tilde{Q}_{0,n}(\xi) &= z^n(1-\xi)\lambda_n^{(3+\xi)/2}[1+\xi+3\lambda_n(1-\xi)+\dots] \\ Q_{1,n}^* &= 4\sqrt{\pi}[Q_{1,n}(a,-x)-Q_{1,n}(a,x)-iQ_{1,n}(-a,-x)+iQ_{1,n}(-a,x)+8(\tilde{Q}_{1,n}(a)-i\tilde{Q}_{1,n}(-a))] \\ Q_{1,n}(a,x) &= z^n(-a-x)\lambda_n^{(5+a+x)/2}[(1+a)(1+x)-2\lambda_n(1+a)(4+x)+\dots] \\ \tilde{Q}_{1,n}(a) &= z^n(1-a)\lambda_n^{(6+a)/2}[1+a-3\lambda_n(1+a)+\dots] \end{aligned} \tag{3.2}$$

We will use the asymptotic form of the roots z_n and find the sum of the series

$$\begin{aligned} J(s,\theta) &= \sum_{n=p+1}^{\infty} z^n(\theta)\lambda_n^s = (-4i\pi)^{-s}z^{p+1}(\theta)\sum_{k=0}^{\infty} z^k(\theta)(\upsilon+k)^{-s}(1+\upsilon_k+\upsilon_k/t_k+\dots)^{-s} = \\ &= (-4i\pi)^{-s}z^{p+1}(\theta)\sum_{k=0}^{\infty} z^k(\theta)(\upsilon+k)^{-s}(1-s\upsilon_k-s\upsilon_k/t_k+s(s+1)\upsilon_k^2/2+\dots) \\ \upsilon &= p+3/4, \quad t_k = 2i\pi(\upsilon+k), \quad \upsilon_k = \ln(4\pi(\upsilon+k))/t_k \end{aligned} \tag{3.3}$$

Differentiation of Euler's integral with respect to the parameter s leads to the relations

$$a_k^{(r)} \equiv z^k(\theta)(\upsilon+k)^{-s}\ln^r(\upsilon+k) = \int_0^\infty t^{s-1}e^{-\upsilon t}K_r(s,t)(z(\theta)e^{-t})^k dt, \quad r = 0, 1, 2$$
$$\Gamma(s)K_0 = 1, \quad \Gamma(s)K_1 = \psi(s) - \ln t, \quad \Gamma(s)K_2 = (\psi(s) - \ln t)^2 - \psi'(s) \tag{3.4}$$

where $\psi(s)$ is the logarithmic derivative of the function $\Gamma(s)$.

Using formula (3.4), we find the following integral respresentations:[7]

$$\Sigma_r \equiv \sum_{k=0}^\infty a_k^{(r)} = \int_0^\infty t^{s-1}e^{-\upsilon t}K_r(s,t)e(\theta,t)dt, \quad e(\theta,t) = (1 - z(\theta)e^{-t})^{-1}, \quad r = 0, 1, 2 \tag{3.5}$$

When $0 < |\theta| < 2$, we expand the function $e(\theta, t)$ in the series

$$e(\theta,t) \equiv (1+\theta_0)[1 - \theta_0(e^{-t}-1)]^{-1} = (1+\theta_0)[1 + \theta_0(e^{-t}-1) + \theta_0^2(e^{-t}-1)^2 + \dots] \tag{3.6}$$

Substituting series (3.6) in the integrand of (3.5) and integrating for $s > 0$, we obtain asymptotic expansions of the quantities Σ_r and $J(s, \theta)$, which hold for large υ and $s > 0$:

$$J(s,\theta) = \frac{(1+\theta_0)z^{p+1}(\theta)}{(-4i\pi\upsilon)^s}\left\{1 - \frac{s\Theta}{\upsilon} + \frac{s[(s+1)(\Theta^2+\theta_0^2+\theta_0) + i\Theta/\pi]}{2\upsilon^2} + O(\upsilon^{-3})\right\}$$
$$\Theta = \theta_0 + \tilde{\upsilon}, \quad \tilde{\upsilon} = \ln(4\pi\upsilon)/(2\pi i), \quad \theta_0 = (i\operatorname{ctg}(\pi\theta/2) - 1)/2, \quad s > 0, \quad 0 < |\theta| < 2 \tag{3.7}$$

Using the series expansion of the function $e(0, t)$, we find the following asymptotic form

$$J(s,0) = (-4i\pi\upsilon)^{-s}[\upsilon/(s-1) - B_1(\tilde{\upsilon}) + i/(2\pi s) + sB_2(\tilde{\upsilon})\upsilon^{-1}/2 + O(\upsilon^{-2})], \quad s > 1 \tag{3.8}$$

where $B_1 = \tilde{\upsilon} - 1/2$ and $B_2 = \tilde{\upsilon}^2 - \tilde{\upsilon} + 1/6$ are Bernoulli polynomials.

Theorem. *Suppose the condition*

$$s(M) > 0, \quad 0 < |\theta(M)| < 2, \quad M \in D$$

is satisfied. Then the functional series (3.3) converges uniformly in D. If, however,

$$s(M) > 1, \quad M \in D_0$$

then the series converges uniformly in D_0 for any values of $\theta(M)$.

Having available the partial sums (3.1) and (3.2) and formulae (3.7) and (3.8), it is possible to investigate the functional series (2.14) for convergence and calculate their residues. In particular, we will ascertain the validity of termwise integration with respect to the variable ξ of the series defining the residue $R_p(\xi, x)$ and examine the procedure for calculating the residue $R_p^{(0)}(x)$. From relations (3.2) for the sum $Q_{0,n}^*$ we obtain the representation

$$Q_{0,n}^* = \sum_{k=0}^N A_k(\xi,x)z^n(\theta_k(\xi,x))\lambda_n^{s_k(\xi,x)}$$

where $s_0(\xi, x) = (2 - \xi - x)/2$ is the smallest exponent.

Let D be a set of points $M(\xi, x)$ such that $\xi, x \in [0; 1]$, $|\xi - x| > 2\rho(\rho \to +0)$. Since on this set the conditions of the theorem are satisfied for all $s_k(M)$ and $\theta_k(M)$ ($k = 0, \dots, N$), the functional series, the sum of which determines the properties of the kernel $K(\xi, x)$, converges uniformly in D. The uniform convergence of the series (2.14) is proved in a similar manner.

Further, using formulae (3.2), (3.7) and (3.8), we calculate

$$R_p(\xi, x) = 2\mathrm{Im}\{E_0(\xi, -x) - E_0(\xi, x) + E_0(-\xi, -x) - E_0(-\xi, x) + 8(\tilde{E}_0(\xi) + \tilde{E}_0(-\xi))\}$$

$$E_0(\xi, x) = (1+\xi)(1+x)J[(2+\xi+x)/2, -\xi-x] +$$
$$+[(1+x)(5-\xi)-(1+\xi)(7+x)]J[(4+\xi+x)/2, -\xi-x] + \dots$$

$$\tilde{E}_0(\xi) = (1+\xi)J[(3+\xi)/2, 1-\xi] + 3(1-\xi)J[(5+\xi)/2, 1-\xi] + \dots$$

$$R_p^{(0)}(x) = 4\sqrt{\pi}\mathrm{Im}\{iE_1(a, -x) - iE_1(a, x) + E_1(-a, -x) - E_1(-a, x) + 8(i\tilde{E}_1(a) + \tilde{E}_1(-a))\}$$

$$E_1(a, x) = (1+a)(1+x)J[(5+a+x)/2, -a-x] -$$
$$-2(1+a)(4+a)J[(7+a+x)/2, -a-x] + \dots$$

$$\tilde{E}_1(a) = (1+a)J[(6+a)/2, 1-a] - 3(1+a)J[(8+a)/2, 1-a] + \dots$$

In order to investigate the kernel $K(\xi, x)$ in the strip $D^*\{|\varepsilon| \le \rho\}$ ($\varepsilon = (\xi - x)/2$), we will examine the truncated formulae

$$\begin{aligned} J(s, \theta) &= (-4i\pi)^{-s} z^{p+1}(\theta)\Sigma_0 = (-4i\pi)^{-s} z^{p+1}(\theta)\Phi(z, s, \upsilon) \quad (z = z(\theta)) \\ E_0(\pm\xi, \mp x) &= (1 \pm 2\varepsilon - x\xi)(-4i\pi)^{-1\mp\varepsilon} z^{p+1}(\mp 2\varepsilon)\Phi[z(\mp 2\varepsilon), 1 \pm \varepsilon, \upsilon] \end{aligned} \tag{3.9}$$

The special function $\Phi(z, s, v)$ can be represented in the form[7]

$$\Phi(z, s, \upsilon) = z^{-\upsilon}\left\{\Gamma(1-s)\ln^{s-1}(1/z) + \sum_{r=0}^{\infty} \varsigma(s-r, \upsilon)\frac{\ln^r z}{r!}\right\}, \quad |\ln z| < 2\pi \tag{3.10}$$

Taking into account relations (3.9) and (3.10), we find the asymptotic form as $\varepsilon \to 0$ of the part of the kernel $K(\xi, x)$ that determines the singularity of $K(\xi, x)$ in the strip D^*:

$$2\mathrm{Im}\{E_0(\xi, -x) - E_0(-\xi, x)\} = \pi^{-1}\varepsilon\ln|\varepsilon|[C_0(1-x\xi)-2] + O(\varepsilon) \ (C_0 = \ln(4\pi) - \psi(1))$$

Further, the functions $E_0(\pm\xi, \pm x)$ and $\tilde{E}_0(\pm\xi)$ are investigated by a similar scheme. As a result it is established that the kernel $K(\xi, x)$ is continuous and bounded in the region $\bar{D}\,\{\xi, x \in [0; 1]\}$, and in the strip D^* has a singularity of the type $(\xi - x)\ln|\xi - x|$. The accuracy of the calculation of the residue R_p was monitored from the quantity ε_p. Thus, by checking the residues $R_p^{(0)}(0.5)$ and $R_p(10, 20)$, the following values are obtained

$$R_p^{(0)}(0.5) = 2.09 \cdot 10^{-12}, \quad \varepsilon_p = 6 \cdot 10^{-20}; \quad R_p(10, 20) = -3.70 \cdot 10^{-21}, \quad \varepsilon_p = 2 \cdot 10^{-29}$$

$$R_p = \sum_{n=p+1}^{\infty} a_n, \quad R = \sum_{n=4001}^{6000} a_n, \quad \varepsilon_p = |r - R|, \quad r = R_{4000} - R_{6000}$$
$$(a = 0.25, b = 0.5, p = 4000)$$

4. Determination of the contact pressure

We will give examples of the calculation for a plane punch ($\delta(x) \equiv \delta_0$, $k = 0$) for the following versions: (1) $a = 0.15$, $b = 0.45$; (2) $a = 0.25$, $b = 0.5$; (3) $a = b = 0.2$. The infinite system (2.17) in the unknowns $Y_r^{(0)}$ ($r = 0, 1, \dots$) (the zero superscript will be omitted below) was truncated and solved for several values of α. For each version, its lowest value of α (equal to 10^{-17}, 10^{-18} and 2×10^{-18} for versions 1, 2 and 3 respectively) for which there were still no appreciable amplitudes of fluctuations of the regularized solution Y_r ($r = 0, \dots, 80$) was chosen, and the discrepancy was sufficiently small ($|\sigma_0(x)| < 10^{-18}$, $a < x \le 1$). Values of the constants $Y_r \times 10^5$ ($r = 0, \dots, 5$; $r = 75, \dots, 80$) are given in Table 1.

Table 1

r	Versions		
	1	2	3
0	−245380	−231506	−373939
1	118683	100002	220430
2	157552	171189	413550
3	5556	−23887	560212
4	−37360	−51764	319277
5	−49377	−57690	224426
75	−295	−642	1776
76	146	265	−2573
77	−288	−638	1558
78	117	192	−2400
79	−297	−664	1362
80	77	102	−2260

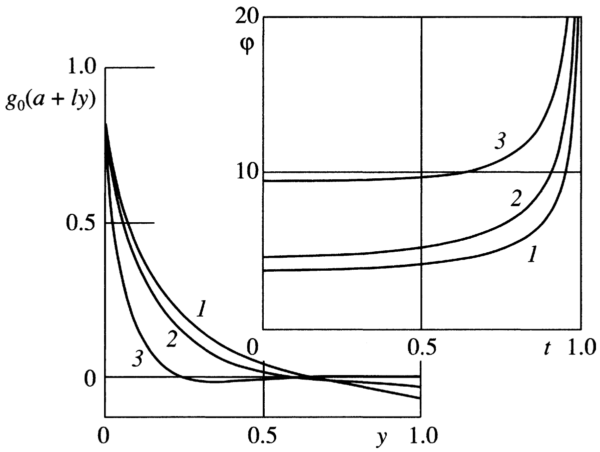

Fig. 2.

Fig. 2 shows graphs of the function $g_0(a+ly)$ $(0 \le y \le 1)$, obtained on the basis of formula (2.18); the number on the curves corresponds to the number of the version.

In order to find the contact pressure $q(x) = -\sigma_y(x, b)$ $(|x| \le a)$, we return to relations (2.19) at $k=0$:

$$\sigma(x) = \theta\delta_0\sigma_0(x), \quad \sigma_0(x) = -\gamma_0 f(x) - \omega(x), \quad \gamma_0 = 1 + \sum_{s=0}^{80} Y_s j_s$$
$$\omega(x) = \sum_{s=0}^{80} Y_s f_s(x), \quad \sigma_y(x, b) = \theta\delta_0\sigma_0'''(x) \quad (\varepsilon_0 = 1,\ f^{(0)}(x) = 0) \tag{4.1}$$

The functions $f_s(x)$, determined by formulae (2.14), allow of the representation

$$f_s(x) = c_s + \tilde{f}_s(x) \tag{4.2}$$

where $c_s = \text{const}$ and $\tilde{f}_s(x)$ are odd functions of x.

Table 2

k	Versions		
	1	2	3
0	4.697	3.853	9.545
1	4.750	3.885	9.551
2	4.926	3.993	9.589
3	5.287	4.223	9.738
4	6.024	4.712	10.237
5	7.928	6.034	12.087

Table 3

λ	1	2	3
χ_1^0	4.709	2.730	2.099
χ_1^*	4.710	2.750	2.104
χ_2^0	1.909	0.963	0.705
χ_2^*	1.905	0.964	0.705
χ_3^0	1.152	0.794	0.648
χ_3^*	1.135	0.795	0.636

As follows from relations (4.1), the dimensionless function of the contact pressure distribution $\tilde{\varphi}(x) = q(x)(\theta\delta_0)^{-1}$ and the indenting force N_0 are defined by the expressions

$$\tilde{\varphi}(x) = -\sigma_0''(x) = 2b^{-1}\gamma_0 + \omega'''(x)$$
$$aN_0 = -2\int_0^a \sigma_0'''(x)dx = -2\sigma_0'''(a) + 4b^{-1}\gamma_0 - 2\sum_{s=0}^{80} Y_s f_s''(0) \tag{4.3}$$

Taking into account representation (4.2) and the equations $\sigma_0''(a) = f_s''(0) = 0$, we find $N_0 = 4\gamma_0(ab)^{-1}$.

We will consider the procedure for the numerical differentiation of the function specified on a uniform grid with step $h = x_{k+1} - x_k$ (x_k are the nodes of the grid). To calculate the third-order derivative $\omega'''(x)$ at the central node $x = x_0$ with respect to seven nodes $x_k = x_0 + kh$ ($k = -3, \ldots, 3$), we use a formula of increased accuracy[1,2]

$$\omega'''(x_0) = h^{-3}\left(\Delta^3\omega_0 + \frac{3}{2}\Delta^4\omega_0 + \frac{1}{4}\Delta^5\omega_0 - \frac{1}{8}\Delta^6\omega_0\right) + O(h^4) \tag{4.4}$$

Expressing the finite differences $\Delta^3\omega_0, \ldots, \Delta^6\omega_0$ in terms of nodal values of the function $\omega(x)$, we obtain

$$\omega'''(x_0) = \frac{1}{8}h^{-3}(\omega_{-3} - 8\omega_{-2} + 13\omega_{-1} - 13\omega_1 + 8\omega_2 - \omega_3), \quad \omega_k = \omega(x_k) \tag{4.5}$$

The optimum step h ($0.0003 \le h \le 0.001$) is selected and the accuracy of formula (4.5) in numerical calculations is conveniently investigated using an *a posteriori* estimate from the rate of decrease of the terms in formula (4.4).

Table 2 gives values of the function $\varphi(t) \equiv \tilde{\varphi}(at)$ ($t = x/a$) for $t = t_k = k/6$, and Table 3 gives values of the quantities $\chi_1^0 = aN_0$, $\chi_2^0 = a\varphi(0)$ and $\chi_3^{(0)} = a \lim \varphi(t)\sqrt{1-t^2}$ ($t \to 1$).

Comparing the values of $\chi_r^0 (r = 1, 2, 3)$ for a rectangle with the corresponding values of χ_r^* for a layer with parameters a and λ ($\lambda = b/a$), lying without friction on a rigid base,[5,8] we see that they differ by less than 1.8% (version 1), 0.7% (version 2) and 1.5% (version 3). Fig. 2 also shows graphs of the function $\varphi(t)$, obtained from formula (4.3).

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